An Accurate 2-D Hard-Source Model for FDTD

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***Abstract*—This letter describes an accurate two-dimensional (2-D) hard-source model for finite-difference time-domain (FDTD). The proposed model allows accurate control over the effective radius of a 2-D hard source. In addition, the TM version of the source model is directly applicable as a very accurate 2-D thin-wire model. The proposed model is verified in 2-D for both TM and TE case using a recently introduced method for finding the effective radius of a filamentary 2-D hard source [1].**

***Index Terms*—FDTD, hard source, thin wire.**

I. Introduction

Two-dimensional (2-D) finite-difference time-domain (FDTD) solvers are used for problems where translational invariance is a reasonable approximation. In 2-D, a single axial field component is typically used to excite a radially outgoing wave. A hard source is realized by forcing the field component to assume the value of the excitation function at each time step. Recently, a method to find the effective radius $r_{\rm eff}$ of a 2-D filamentary hard source was presented [1]. By comparison to an analytical frequency-domain Green's function, it was shown that $r_{\rm eff} \approx 0.2\Delta$ where Δ is the cell size. Size of the source thus depends on the cell size. However, the size of the source affects the simulation results. It is therefore important to have control over the size of the source.

A filamentary perfectly conducting wire can be understood as a hard source with the excitation function identically set to zero. It is obvious that in a 2-D TM case, the procedure of [1] can also be applied to find the effective radius of a perfectly conducting filamentary thin wire or a special thin-wire model. In other words, [1] is a useful 2-D test also for thin-wire models.

This letter describes a sub-cell FDTD model based on contour-path integration that accurately describes the radius of a 2-D hard source or a 2-D thin wire over a wide range of values for nominal radius.

II. Contour-Path Integral Formulation of FDTD

The contour-path integral equation formulation is a crucial part of the proposed model and is therefore briefly reviewed. In the Yee algorithm [2], the electric field components form the primary grid while the magnetic field components form the dual grid. The two grids are organized such that each edge of the primary grid is normal to a facet of the dual grid and vice versa. The respective location of primary and dual grids allows us to enforce Faraday's law and Ampere's law in integral form over each facet of primary and dual grid, respectively. The Ampere's law in free space is given by

$$\frac{\partial}{\partial t}\int_{S_d}\varepsilon_0\mathbf{E}\cdot d\mathbf{S}_d=\oint_{C_d}\mathbf{H}\cdot d\mathbf{l}_d \tag{1}$$

where surface S_d is a dual grid facet bounded by contour C_d formed by dual grid edges. Similarly, the Faraday's law in free space is given by

$$-\frac{\partial}{\partial t}\int_{S_p}\mu_0\mathbf{H}\cdot d\mathbf{S}_p=\oint_{C_p}\mathbf{E}\cdot d\mathbf{l}_p \tag{2}$$

where surface S_p is a primary grid facet bounded by contour C_p formed by primary grid edges.

The contour-path integral formulation in Cartesian grid and the original Yee algorithm are equivalent in free space, provided the following assumptions are met [3]:

1) the field value at a midpoint of an edge along contour C enclosing surface S equals the average value of that field component along that edge;
2) the field value normal to surface S is located at the center of the surface and equals the average value of that field component over S.

III. Accurate 2-D Hard-Source Model

The assumptions on field variation near the wire are similar to those in [3]: both the electric field in TE mode and the magnetic field in TM mode adjacent to source are assumed $1/r$ dependent, where r is distance from the wire axis. The field components are loacted at their usual locations in a 2-D Yee cell.

A. Derivation for TM Mode

Let us consider computation of one of the four magnetic field components adjacent to an axial electric source at $(i+1/2,\ j+1/2)$. The update equation for the magnetic field at $(i+1,\ j+1/2)$ can be found using equation (2) in 2-D. Assuming $1/r$ dependence, we have in free space

$$H_\theta|_{i+1,\,j+1/2}^{n+1/2}=H_\theta|_{i+1,\,j+1/2}^{n-1/2}+\frac{2\Delta t}{\mu_0\Delta x\ln(\Delta x/r_i)}\cdot\left[E_z|_{i+1/2,\,j+1/2}^{n}-E_z|_{i+3/2,\,j+1/2}^{n}\right], \tag{3}$$

where r_i is the nominal radius of the electric source.

Due to the $1/r$ assumption the computed magnetic field components represent tangential magnetic field along a circular path around the source as shown in Fig. 1(a). They are point values at distance $\Delta/2$ from the source and do not represent the average value of the field component along a Cartesian dual grid edge as required in Assumption 1 in Section II. Therefore, the

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field component should not be directly used in standard FDTD update equations. Therefore, we need to project the tangential field components to their respective Cartesian grid edges.

Let us assume we have computed any of the $H_{\theta i}$ in Fig. 1(a) using (3). In order to project $H_{\theta i}$ into $k_{\rm TM} H_{\theta i}$ on a Cartesian grid edge we take Ampere's law as a starting point. Forming two integration paths C_1 and C_2 around the source, we have

$$I = \oint_{C_1} \mathbf{H} \cdot d\mathbf{l}_1 = \oint_{C_2} \mathbf{H} \cdot d\mathbf{l}_2. \tag{4}$$

For a rotationally symmetric case, the magnetic field is constant along C_1. The scaling factor $k_{\rm TM}$ is determined by the relative length of integration paths C_1 and C_2

$$k_{\rm TM} = \oint_{C_1} d\mathbf{l}_1 \Big/ \oint_{C_2} d\mathbf{l}_2 = \pi/4. \tag{5}$$

In a nonsymmetric case, the four looping magnetic field values may differ from each other. Still, we may define a scaling factor for each segment such that the line integral of $\mathbf{H}$ along each circular segment remains equal to the line integral of $\mathbf{H}$ along the respective Cartesian dual grid edge

$$k_{{\rm TM},\, i} = \int_{C_{1i}} d\mathbf{l}_1 \Big/ \int_{C_{2i}} d\mathbf{l}_2 \,. \tag{6}$$

Here, we assume $H_{\theta i}$ and $k_{\rm TM} H_{\theta i}$ constant on each of the four segments of C_1 and C_2, respectively. Note that the shape of the cells affects the value of $k_{{\rm TM},\, i}$. If $\Delta x \neq \Delta y$, e.g., the scaling factor $k_{{\rm TM},\, 1}$ for the magnetic field component H_y on the right takes the form $(\Delta x/\Delta y) \cdot \tan^{-1}(\Delta y/\Delta x)$.

Instead of scaling the $H_{\theta i}$ themselves, we may use the scaled value $k_{\rm TM} H_{\theta i}$ in standard FDTD update equations to compute the adjacent axial electric field components.

B. Derivation for TE Mode

The TE case is treated similarly to the TM case. Let us consider computation of one of the four electric field components adjacent to a magnetic source at $(i,\, j)$. Assuming the electric field $1/r$ dependent, the update equation for the electric field at $(i + 1/2,\, j)$ can be found using (1) in 2-D. The FDTD update equation in free space takes the form

$$E_\theta|^{n+1}_{i+1/2,\, j} = E_\theta|^{n}_{i+1/2,\, j} + \frac{2\Delta t}{\varepsilon_0 \Delta x \ln(\Delta x / r_i)} \cdot \left[-H_z|^{n+1/2}_{i,\, j+1} + H_z|^{n+1/2}_{i,\, j} \right], \tag{7}$$

where r_i is the nominal radius of the magnetic source.

As before, we project the point values computed using (7) to the Cartesian primary grid edges. Faraday's law integration path is formed around the magnetic source and the circulation of electric field around the source is kept constant. This leads to the scaling factor $k_{TE,\, i}$ for TE mode

$$k_{TE,\, i} = \int_{C_{1i}} d\mathbf{l}_1 \Big/ \int_{C_{2i}} d\mathbf{l}_2 \,. \tag{8}$$

As in TM mode, we do not scale the values themselves, but use the scaled values to compute the adjacent axial magnetic field components.

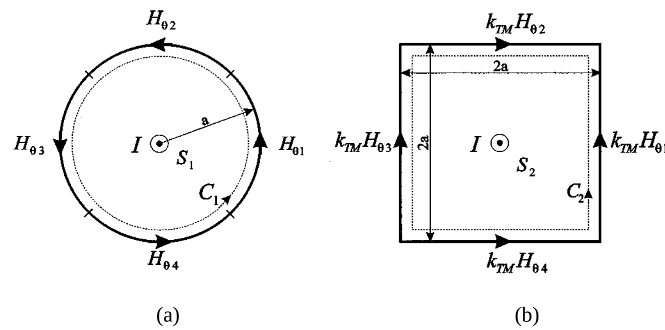

Fig. 1. Ampere's law integration paths. (a) Circular path. (b) Cartesian cell edges.

TABLE I
EFFECTIVE RADIUS OF THE PROPOSED MODEL AND THE THIN-WIRE MODEL [3]. COMPUTED USING A UNIFORM GRID WITH $\lambda/\Delta = 40$ RESOLUTION. TM MODE

	Proposed model		Thin wire [3]	
r_i/Δ	r_{eff} [1]	error %	r_{eff} [1]	error %
1E-6	1.29E-6	29	23.3E-6	2228
1E-3	1.10E-3	9.6	4.7E-03	369
0.05	0.0512	2.4	0.0966	93
0.1	0.1015	1.5	0.1656	66
0.2	0.2016	0.8	0.2840	42
0.3	0.3014	0.5	0.3892	30
0.4	0.4006	0.2	0.4871	22
0.49	0.4902	<0.1	0.5871	20

IV. NUMERICAL RESULTS AND DISCUSSION

The source models were tested following the procedure described in [1] using the maximum absolute value for amplitude recorded at each point. The test reveals properties of the source when observed from a distance. It is likely that the accuracy of the test method is dependent on the grid resolution providing more accurate results with a dense grid due to smaller dispersion error.

The proposed model was also tested using two independent tests. The current through a hard source was computed for several grid resolutions such that neither the physical dimensions of the source nor the spectrum of the excitation waveform were changed. In addition, current induced into a passive thin wire was computed for several grid resolutions.

A. The 2-D Green's Function Test

The proposed model was tested using [1] for both TM mode and TE mode. A time-harmonic hard source with unity amplitude was used for excitation. After reaching steady state, the envelope of the response was recorded at each grid point along a coordinate axis providing amplitude as function of distance from the source. The collected data was then multiplied by a constant C such that the best fit to the absolute value of analytical frequency-domain Green's function solution (in this case, Hankel function of the second kind) was achieved in the L_2 norm sense. The Hankel function at distance $r_{\rm eff}$ from the source was set equal to C and $r_{\rm eff}$ appearing in the argument was solved. The effective radius $r_{\rm eff}$ describes a finite distance from the source to the location where the wave is actually radiating.

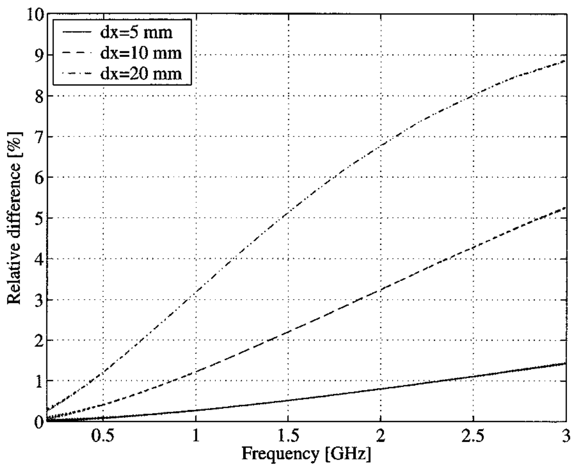

Fig. 2. Current through a TM hard source implemented using the proposed model for a number of cell sizes, compared to a reference result.

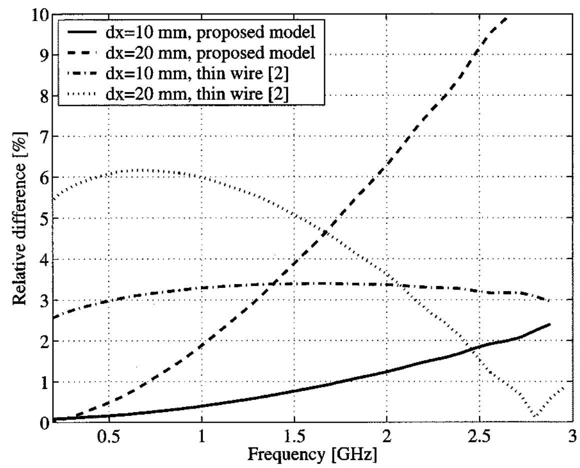

Fig. 3. Relative difference of normalized induced currents on a passive thin wire for a number of cell sizes, compared to a reference result.

The results computed using 40 cells per λ resolution are shown in Table I. The error in source radius with the proposed model is less than 30% for radii from $10^{-6}\Delta$ up to the stability limit of 0.5Δ. For radii larger than 0.05Δ, the error is less than 2.5%. The same test was repeated using the standard thin-wire model [2]. The results in Table I indicate that the error in wire radius is over ten times larger than with the proposed model.

A source model should reduce to a filamentary source when the nominal radius is set to the effective radius of a filamentary source. With $\Delta x = \Delta y = \Delta$, the proposed model reduces to a filamentary hard source when the nominal radius is set to $e^{-\pi/2}\Delta \approx 0.208\Delta$. This is in agreement with [1], where the effective radius of a filamentary source was found to be approximately 0.21Δ. In contrast, the three–dimensional thin-wire model of [3] reduces to a filamentary PEC wire when the nominal wire radius is set to $e^{-2}\Delta \approx 0.135\Delta$ [4]. This is also the case in 2-D TM mode. This is due to the fact that the magnetic field components adjacent to wire receiving special treatment in [3] are also present in 2-D TM mode. It is therefore likely that a similar error as shown in Table I is also present in the 3-D model of [3].

The results in TE mode (not shown) are identical to those presented for TM mode in Table I verifying the TE source model. Furthermore, the proposed model reduces to a filamentary source also in TE mode when the nominal radius is set to $e^{-\pi/2}\Delta \approx 0.208\Delta$, i.e., the effective radius of a filamentary source.

B. Current Through a TM Hard Source

The proposed model was tested in TM mode by computing the current through a hard source. Uniform grids with 3.33 mm, 5 mm, 10 mm, and 20 mm cells were considered with source radius fixed at 1 mm. The current computed using the smallest cell size 3.33 mm was chosen as a reference. The results are shown in Fig. 2. The currents computed using different cell sizes converge to the same value as we move toward dc, i.e., the resolution is improved. This indicates that the accuracy of the proposed model is improved as the mesh density is increased.

C. Proposed Model as a Passive Structure

As a complementary test, the properties of the proposed model as a passive scattering structure in TM mode were investigated. A point source was excited at a fixed distance from the thin wire, whose radius was fixed at 1 mm. Cell sizes 5 mm, 10 mm and 20 mm were considered. The induced current was measured and normalized by the square root of the radiated power of the source. The induced current with the 5 mm cells was used as a reference. For comparison, the same test was repeated using the traditional thin-wire model [2]. The results are shown in Fig. 3. Obviously, the proposed model gives consistent results independent of the cell size for adequate grid resolution, while the traditional model does not.

V. Conclusion

An accurate 2-D hard-source model was proposed. The model was validated for both TM mode and TE mode using the procedure described in [1]. The proposed model was shown to provide accurate results over a wide range of values for the nominal wire radius.

Additional tests investigating current through the source in TM mode and the current induced in a passive thin wire were provided for further verification. The proposed model was shown to provide consistent results independent of the cell size both as a hard source and as a passive 2-D thin-wire model.

References

[1] G. Waldschmidt and A. Taflove, “The determination of the effective radius of a filamentary source in the FDTD mesh,” *IEEE Microwave Guided Wave Lett.*, vol. 10, pp. 217–219, June 2000.

[2] K. S. Yee, “Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media,” *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, May 1966.

[3] A. Taflove, K. Umashankar, B. Beker, F. Harfoush, and K. Yee, “Detailed FD-TD analysis of electromagnetic fields penetrating narrow slots and lapped joints in thick conducting screens,” *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 247–257, Feb. 1988.

[4] S. Watanabe and M. Taki, “An improved FDTD model for the feeding Gap of a thin-wire antenna,” *IEEE Microwave Guided Wave Lett.*, vol. 8, no. 4, pp. 152–154, Apr. 1998.